LIMIT SETS AND CLOSED SETS IN SEPARABLE METRIC SPACES

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Abstract. Example 2 in Chapter 5 of [1] constructs, for an arbitrary closed subset of the real line, a sequence whose set of limit points is exactly the original closed set. We use a similar construction to show that an arbitrary nonempty closed set in a separable metric space is always the set of limit points of some sequence. We note further that if all nonempty closed subsets of a metric space can be realized as sets of limit points of sequences, then that metric space is separable.

A subset A of the real line is closed if and only if there is a sequence of real numbers with the property that A is precisely the set of limits of convergent subsequences of that sequence. One construction of such a sequence for an arbitrary closed set A is found in [1]. We were first made aware of this result by Tom Sibley [3]. The purpose of this note is to determine which other metric spaces have this property. We have attempted to provide enough background and detail in this note to make it accessible to an undergraduate with some basic knowledge of real analysis and metric space topology.

We use the notation (x_n) to denote a sequence and $\{x_n\}$ to denote the set of points of the sequence. The limit of a convergent subsequence of a sequence is a *subsequential limit* of the sequence; the set of all subsequential limits is the *limit set* of the sequence. If A is a subset of the space X, we use $\overline{A}$ to denote the closure of A. Finally, if $S = (x_n)$ is a sequence we use $L(S)$ or $L(x_n)$ to denote the limit set of S.

We will make use of the following well-known result [2].

Theorem 1. Let X be a metric space and (x_n) a sequence of points in X. If p is a limit point of $\{x_n\}$, then there is a subsequence of (x_n) that converges to p.

The converse of the above theorem is not true. For example, the sequence $(1, 2, 1, 3, 1, 4, 1, 5, \dots)$ has a constant subsequence that converges to 1, which is not a limit point of the set $\{1, 2, 3, \dots\}$.

We are interested in two questions.

1. For which metric spaces is the limit set of every sequence closed?
2. For which metric spaces is every nonempty closed set the limit set of a sequence?

The first question is answered by the following theorem.

Theorem 2. The limit set of a sequence in a metric space is closed.

Proof. Let (x_n) be a sequence in the metric space X and let A be the limit set of (x_n). Suppose that p is a limit point of A and let U be an open neighborhood of p, then U must contain a point a of A distinct from p. Since X is Hausdorff, there are disjoint open neighborhoods V of a and W of p such that $V \cup W \subset U$. The point a is a subsequential limit of (x_n), so V must contain some term x_k of the sequence (x_n). Since $V \cap W = \emptyset$ and $V \subset U$, $x_k \neq p$ and $x_k \in U$. We have shown that every open neighborhood U of p contains a point of $\{x_k\}$ distinct from p, so p is a limit point of the set $\{x_n\}$. By Theorem 1, p must be a subsequential limit of (x_n), hence in A. Since A contains all of its limit points, A is closed.

The remainder of this note is devoted to answering the second question. Note first that if X is a metric space in which every closed set is the limit set of a sequence, then X itself is the limit set of some sequence (x_n). It is routine to show that the set $\{x_n\}$ is dense in X, so X must be separable. We will show that in any separable metric space, every nonempty closed set is the limit set of a sequence. We note that our general result differs from that for the real line in that we are only able to realize nonempty closed sets as limit sets of sequences. This additional hypothesis is critical when the metric space in question is compact since every sequence in a compact metric space has a convergent subsequence.

The following elementary result concerning the algebra of limit sets will prove useful.

Theorem 3. Let X be a metric space and let A and B be limit sets of sequences in X, then $A \cup B$ is the limit set of a sequence in X.

Proof. Suppose that A is the limit set of the sequence (x_n) and that B is the limit set of the sequence (y_n). We create a new sequence (z_n) by "shuffling" these two sequences. More precisely, define $z_{2k} = x_k$ and $z_{2k+1} = y_k$. We claim that $A \cup B$ is the limit set of (z_n). Note first that if $p \in A$, then p is the limit of some subsequence (x_{n_k}) of (x_n). This subsequence is also a subsequence of (z_n), so p is in the limit set of (z_n). A similar argument shows that every element of B is in the limit set of (z_n), so $A \cup B$ is a subset of the limit set of (z_n). To see that the opposite containment holds, suppose that q is the limit of a subsequence (z_{n_k}) of (z_n). Considering our construction of (z_n), the subsequence (z_{n_k}) must contain a subsequence (w_i) of either (x_n) or (y_n). Since $z_{n_k} \to q$, it must be true that $w_i \to q$. Hence, either $q \in A$ or $q \in B$ as desired.

Suppose now that X is a separable metric space. We wish to construct, for any closed set A in X, a sequence with limit set A. The existence of isolated points (a point x is isolated if the singleton $\{x\}$ is open in X) in A changes our construction, so we choose to deal with those points separately. Since our space X is separable, and since any dense subset of X must contain every isolated point, there can be at most countably many isolated points in X.

For any point $p \in X$, the constant sequence (p) has limit set $\{p\}$, so every singleton is the limit set of some sequence. Applying Theorem 3, we see that any finite set is the limit set of a sequence. If $A = \{p_1, p_2, p_3, \dots\}$ is a countably infinite subset of X and each point of A is an isolated point, consider the sequence S constructed by arranging the points of A in the following order: $S = (p_1, p_2, p_1, p_2, p_3, p_1, p_2, p_3, p_4, p_1, \dots)$. We note two important facts about S:

1. For each point of A, there is a (constant) subsequence of S converging to that point.
2. Every sequence of points in A is a subsequence of S.

These facts imply that $A \subseteq L(S) \subseteq \overline{A}$. Since the limit set of S is closed, it follows that $L(S) = \overline{A}$.

We now have the following theorem.

Theorem 4. Let X be a separable metric space and A a nonempty subset of X such that every point of A is an isolated point of X. Then there is a sequence of points in X with limit set $\overline{A}$.

For a closed set containing no isolated points we have the following theorem.

Theorem 5. Let X be a separable metric space and A a nonempty closed subset of X such that no point of A is an isolated point of X. Then there is a sequence (x_n) in X so that A is the limit set of (x_n).

Proof. Let d denote the metric on X.

Since X is a separable metric space, there is a countable basis $\{B_1, B_2, \dots\}$ for the topology on X. Fix a point $a_0 \in A$. We construct the desired sequence by choosing terms from each basis element that intersects A. Since we want points of A to be limit points of our sequence, we find it convenient to choose two distinct terms from each such basis element. To be more precise, we choose x_{2n-1} and x_{2n} as follows:

If $A \cap B_n = \emptyset$, then $x_{2n-1} = x_{2n} = a_0$.

If $A \cap B_n \neq \emptyset$, choose distinct points x_{2n-1} and $x_{2n} \in B_n$ with $d(x_{2n-1}, A) < 1/n$ and $d(x_{2n}, A) < 1/n$. This is always possible since no point of A is an isolated point, so every open set containing a point of A must contain at least two distinct points.

We will show that A is the limit set of the sequence (x_n). If $a \in A$ and B_n is a basis element containing a, then x_{2n-1} and x_{2n} are distinct points of $\{x_n\}$ in B_n. Since at least one of these points must be distinct from a, B_n contains a point of $\{x_n\}$ other than a. Hence, a is a limit point of $\{x_n\}$ and must be a subsequential limit of the sequence (x_n) by Theorem 1. Therefore, $A \subseteq L(x_n)$.

Now suppose that $w \notin A$. Since A is closed, there must exist $\epsilon > 0$ so that $d(x, w) < \epsilon$ implies that $x \notin A$. Choose a natural number K large enough that $2/K < \epsilon/2$. Finally, note that for any subsequence (z_k) of (x_n) there is an index J such that if $j \geq J$ and $z_j = x_m$, then $m \geq K$. We will show that for any $j \geq J$, $d(z_j, w) \geq \epsilon/2$, so no subsequence of (z_k) converges to w. Assume to the contrary that $d(z_j, w) < \epsilon/2$ for some $j \geq J$. By our choice of J, $z_j = x_m$ for some $m \geq K$. If $m = 2n$, then there is an $a \in A$ so that $d(a, x_m) = d(a, x_{2n}) < 1/n = 2/m$. If $m = 2n - 1$, then there is an $a \in A$ so that $d(a, x_m) = d(a, x_{2n-1} < 1/n = 2/(m+1) < 2/m$. In either case we have

$$d(a, w) \leq d(a, x_m) + d(x_m, w) < 2/m + \epsilon/2 < \epsilon/2 + \epsilon/2 = \epsilon, \qquad (1)$$

which contradicts our choice of ϵ.

We are now ready to state and prove the desired result.

Theorem 6. Let X be a separable metric space and A a nonempty closed subset of X. Then there is a sequence (x_n) in X so that A is the limit set of (x_n).

Proof. Let B be the set of isolated points in A and let C be the remaining points in A. Since B is a union of open singletons, B is open. Thus, $C = A - B$ is closed and Theorem 5 implies that C is the limit set of a sequence in X. Furthermore, Theorem 4 implies that $\overline{B}$ is the limit set of a sequence in X, so $\overline{B} \cup C$ is the limit set of a sequence by Theorem 3. Since A is closed, $\overline{B} \subseteq A$. Hence,

$$A = B \cup C \subseteq \overline{B} \cup C \subseteq A, \qquad (2)$$

and $A = \overline{B} \cup C$ is the limit set of a sequence as desired.

References

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