

ABSTRACT

On the Natural Density of the Range
of the Terminating Nines Function

by

Curtis N. Cooper
Robert E. Kennedy

Central Missouri State University
Warrensburg, Missouri

Noting that the expression

$$\sum_{t \geq 1} \left[\frac{n}{10^t} \right]$$

gives the number of terminating nines which occur up to n but not including n , we will denote the above expression by $t(n)$ and call t the "terminating nines function." The natural density of the set $T = \{ t(n) : n = 1, 2, 3, \dots \}$ will be determined.

AMS (MOS) subject classifications (1985) . Primary: 11A63 , Secondary: none.
Key words and phrases. Digital sums, terminating nines, natural density.

Robert E. Kennedy
Curtis N. Cooper
Department of Mathematics and Computer Science
Central Missouri State University
Warrensburg, Missouri 64093

On the Natural Density of the Range
of the Terminating Nines Function

1. Introduction. The number of positive integers in a set A , not exceeding x , is denoted by $A(x)$. The natural density, $d(A)$, of the set A is defined as

$$d(A) = \lim_{x \rightarrow \infty} \frac{A(x)}{x} ,$$

provided this limit exists. The determination of the natural density of a given set of positive integers is an important topic in most number theory textbooks and is the subject of much research.

For example, the set of positive integers

$$N = \{ n : s(n) \text{ is a factor of } n \} ,$$

where $s(n)$ denotes the digital sum of n , is the set of Niven numbers [1] and was shown to have a natural density of 0 in [2] . Here, we are interested in a part of the digital sum function.

It has been shown that

$$s(n) = n - 9 \sum_{t \geq 1} \left[\frac{n}{10^t} \right]$$

where, as usual, the square brackets denote the integral part operator. Noting that the expression

$$(1) \quad \sum_{t \geq 1} \left[\frac{n}{10^t} \right]$$

gives the number of terminating nines which occur up to n but not including n , we will denote (1) by $t(n)$ and call t the "terminating nines function." The natural density of

the set $T = \{ t(n) : n = 1, 2, 3, \dots \}$ will be determined in what follows. Note that T does not include every positive integer since, for example, $10 \notin T$.

2. Notation and Terminology. In what follows, we will say that the terminating nines function, t , has a "jump" of size k at an integer a if $t(a) = t(a - 1) + k$. Thus, t has a jump of size k if and only if $a - 1$ ends with exactly k nines. To determine the natural density of T , we first show that

$$\lim_{n \rightarrow \infty} \frac{T(t(n))}{t(n)} = \frac{9}{10},$$

where $T(t(n))$ is the number of members of T not exceeding $t(n)$. To do this, we will count how many integers are missing from the set $\{ t(1), t(2), \dots, t(n) \}$. If a_n is the number of these missing integers, then it follows that

$$T(t(n)) = t(n) - a_n.$$

3. The Natural Density of T . Noting that if $1 \leq a \leq n$ and t has a jump of size k at a , then this jump will produce $k - 1$ missing integers. Moreover, each missing integer is a result of some jump at a for $1 \leq a \leq n$. Thus, each $1 \leq a \leq n$, such that 10^k divides a but 10^{k+1} does not divide a , produces $k - 1$ missing integers. Hence, a_n is the number of terminating 0's in all integers $1 \leq a \leq n$, minus the number of integers $1 \leq a \leq n$ which end with 0. Therefore, since

$$a_n = \sum_{j \geq 1} \left[\frac{n}{10^j} \right] - \left[\frac{n}{10} \right],$$

we have that

$$T(t(n)) = \left[\frac{n}{10} \right].$$

Using the above, we thus conclude that

$$\frac{T(t(n))}{t(n)} = \frac{\left[\frac{n}{10} \right]}{\left[\frac{n}{10} \right] + \left[\frac{n}{10^2} \right] + \dots},$$

which may be written as

$$\frac{T(t(n))}{t(n)} = \frac{\frac{n}{10} + O(1)}{\frac{n}{10} + \frac{n}{10^2} + \cdots + O(\log n)},$$

since the denominator is equal $\frac{n}{10} + \frac{n}{10^2} + \cdots + O(\log n)$, and the numerator is equal

to $\frac{n}{10} + O(1)$. Thus,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{T(t(n))}{t(n)} &= \lim_{n \rightarrow \infty} \frac{\frac{n}{10} + O(1)}{\frac{n}{10} + \frac{n}{10^2} + \cdots + O(\log n)} \\ &= \frac{9}{10}. \end{aligned}$$

Letting x be an arbitrary integer, and y be such that

$$t(y) \leq x < t(y + 1),$$

we have that $x - t(y) = O(\log x)$ since $x - t(y)$ does not exceed the number of digits in x .

Since, $T(x) = T(t(y))$, we have

$$\frac{T(x)}{x} = \frac{T(t(y))}{x} = \frac{T(t(y))}{t(y) + O(\log x)}$$

and so, by the above limit, it follows that

$$\lim_{x \rightarrow \infty} \frac{T(x)}{x} = \frac{9}{10}.$$

Stating this as a theorem we have:

Theorem 1. Let $T = \{ t(n) : n = 1, 2, \dots \}$ where t is the terminating nines function. Then $d(T) = \frac{9}{10}$.

5. Generalization to Base b . Finally, it should be noted that the development given above and Theorem 1 can be generalized to any integral base b . If $t_b(n)$ denotes the number of terminating $b-1$'s in the base b representation of the sequence of positive integers up to n , then we have the following generalization of Theorem 1:

Theorem 1'. Let $T = \{ t_b(n) : n = 1, 2, \dots \}$. Then $d(T) = \frac{b-1}{b}$.

References

- [1] R. Kennedy, T. Goodman, and C. Best, Mathematical discovery and Niven numbers, the MATYC Journal 14(1980), 21-25.
- [2] R. Kennedy, C. Cooper, On the natural density of the Niven numbers, College Math. Journal 15(1984), 309-312.

Robert E. Kennedy
Curtis N. Cooper
Department of Mathematics and Computer Science
Central Missouri State University
Warrensburg, MO 64093