ON CERTAIN SUMS OF FUNCTIONS OF BASE B EXPANSIONS

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0. Introduction. Let $s_b(i)$ denote the base 10 sum of the digits in the base b representation of the nonnegative integer i and $L_b(i)$ denote the number of large digits ($\lceil b/2 \rceil$ or more) in the base b representation of the nonnegative integer i. For example, $s_{10}(4567) = 22$, $s_7(7079) = 17$ since $7079 = 26432_7$, and $s_2(19) = 3$ since $19 = 10011_2$. In addition, $L_{10}(4567) = 3$, $L_7(7079) = 2$, and $L_2(19) = 3$. The mathematical literature has many instances of sums involving s_b and L_b. Bush [1] showed that

$$\frac{1}{x} \sum_{n<x} s_b(n) \sim \frac{b-1}{2\log b} \log x.$$

Here, $\log x$ denotes the natural logarithm of x. Mirsky [7], and later Cheo and Yien [2], proved that

$$\frac{1}{x} \sum_{n<x} s_b(n) = \frac{b-1}{2\log b} \log x + O(1).$$

Trollope [9] discovered the following result. Let $g(x)$ be periodic of period one and defined on $[0,1]$ by

$$g(x) = \begin{cases} \frac{1}{2}x, & 0 \le x \le \frac{1}{2} \\ \frac{1}{2}(1-x), & \frac{1}{2} < x \le 1, \end{cases}$$

and let

$$f(x) = \sum_{i=0}^{\infty} \frac{1}{2^i} g(2^i x).$$

Now, if $n = 2^m(1+x)$, $0 \le x < 1$, then

$$\sum_{i<n} s_2(i) = \frac{1}{2\log 2} n \log n - E_2(n),$$

where

$$E_2(n) = 2^{m-1} \left\{ 2f(x) + (1+x)\frac{\log(1+x)}{\log 2} - 2x \right\}.$$

In addition, it was shown in [6] that

$$\sum_{i=1}^{\infty} \frac{L_{10}(2^i)}{2^i} = \frac{2}{9}.$$

We will discuss some other sums involving s_b and L_b. In particular, we will give formulas for

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(L_b(i))^m \quad\text{and}\quad \frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i))^m,$$

where m and n are positive integers. Then, we will find a formula for

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1} s_b(i)\cdot L_b(i).$$

We define

$$C_b(x;\ y)$$

to be the sum of the carries when the positive integer x is multiplied by y, using the normal multiplication algorithm in base b arithmetic. That is, we convert x and y to base b and then multiply in base b. In this algorithm, we consider the carries above the numbers as well as in the columns. We will prove that

$$\sum_{i=1}^{\infty}\frac{C_b(a;\ a^i)}{(s_b(a))^i}=\frac{s_b(a)}{b-1}.$$

We will conclude the paper with some open questions.

1. First Sum. To compute

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(L_b(i))^m,$$

we begin with the function

$$f(x)=(\underbrace{1+\cdots+1}_{\lceil b/2\rceil \text{ times}}+\underbrace{e^x+\cdots+e^x}_{\lfloor b/2\rfloor \text{ times}})^n=(\lceil b/2\rceil+\lfloor b/2\rfloor e^x)^n.$$

The motivation for this function comes from the fact that in the base b representation of $i = i_n\cdots i_2 i_1$, the jth digit of i, i_j, is either small or large and thus contributes 0 or 1 to the number of large digits in i. Expanding the product, we see that there is a 1-1 correspondence between the numbers $0\le i\le b^n-1$ and the b^n terms $1\cdot e^{L_b(i)x}$. Therefore,

$$f(x)=(\lceil b/2\rceil+\lfloor b/2\rfloor e^x)^n=\sum_{i=0}^{b^n-1}1\cdot e^{L_b(i)x}.$$

Thus,

$$f^{(m)}(x) = \sum_{i=0}^{b^n-1} (L_b(i))^m e^{L_b(i)x},$$

and so we have that

$$f^{(m)}(0) = \sum_{i=0}^{b^n-1} (L_b(i))^m.$$

To continue our discussion, we need the idea of Stirling numbers of the first and second kinds. A discourse on this subject can be found in [3]. A Stirling number of the second kind, denoted by

$$\left\{ {n \atop k} \right\},$$

symbolizes the number of ways to partition a set of n things into k nonempty subsets. A Stirling number of the first kind, denoted by

$$\left[{n \atop k} \right],$$

counts the number of ways to arrange n objects into k cycles. These cycles are cyclic arrangements of the objects. We will use the notation $[A, B, C, D]$ to denote a clockwise arrangement of the 4 objects A, B, C, D in a circle. For example, there are eleven different ways to make two cycles from four elements,

$$\begin{array}{llll} [1,2,3][4], & [1,2,4][3], & [1,3,4][2], & [2,3,4][1], \\ [1,3,2][4], & [1,4,2][3], & [1,4,3][2], & [2,4,3][1], \\ [1,2][3,4], & [1,3][2,4], & [1,4][2,3]. & \end{array}$$

Hence,

$$\left[{4 \atop 2} \right] = 11.$$

Now it can be shown, by induction on m, that

$$f^{(m)}(x) = \sum_{j=1}^{m} \left\{ {m \atop j} \right\} n^{\underline{j}} (\lfloor b/2 \rfloor e^x)^j (\lceil b/2 \rceil + \lfloor b/2 \rfloor e^x)^{n-j},$$

where

$$n^{\underline{j}} = n(n-1)\cdots(n-j+1).$$

The last quantity is known as the jth falling factorial of n. A discussion of this idea can be found in [3]. Thus,

$$\sum_{i=0}^{b^n-1}(L_b(i))^m = \sum_{j=1}^{m}\left\{{m \atop j}\right\} n^{\underline{j}}\lfloor b/2\rfloor^j \cdot b^{n-j} = b^n\sum_{j=1}^{m}\left\{{m \atop j}\right\}\left(\frac{\lfloor b/2\rfloor}{b}\right)^j n^{\underline{j}}.$$

Since

$$n^{\underline{j}} = j!\binom{n}{j},$$

we have proved the following theorem.

Theorem 1. Let m and n be nonnegative integers. Then

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(L_b(i))^m = \sum_{j=1}^{m}\left\{{m \atop j}\right\}\left(\frac{\lfloor b/2\rfloor}{b}\right)^j \cdot j!\binom{n}{j}.$$

To illustrate this theorem, if $b = 5$, $m = 3$, and n is a nonnegative integer, then

$$\frac{1}{5^n}\sum_{i=0}^{5^n-1}(L_5(i))^3 = \frac{8}{125}n^3 + \frac{36}{125}n^2 + \frac{6}{125}n.$$

2. Second Sum. Let m and n be positive integers. The determination of the sum

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1}(s_{10}(i))^m$$

was an open question in [4]. In [10], David Zeitlin presented the following answer to the problem in base 10. He stated that if

$$B_i^{(n)}$$

denotes Bernoulli numbers of order n, where

$$\binom{n-1}{i}\cdot B_i^{(n)} = \begin{bmatrix} n \\ n-i \end{bmatrix},$$

then

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1}(s_{10}(i))^m = \binom{n+m}{m}^{-1}\sum_{i=0}^{m}10^i\cdot\binom{n+m}{m-i}\left\{{n+i \atop n}\right\}\cdot B_{m-i}^{(n)}.$$

To compute

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i))^m,$$

we make use of the function

$$(g(x))^n,$$

where

$$g(x)=1+e^x+e^{2x}+\cdots+e^{(b-1)x}.$$

The motivation for this function comes from the fact that in the base b representation of $i=i_n\cdots i_2i_1$, the jth digit of i, i_j, contributes i_j to the digital sum of i. Expanding the product, we see that there is a 1-1 correspondence between the numbers $0\le i\le b^n-1$ and the b^n terms $1\cdot e^{s_b(i)x}$. Therefore,

$$(g(x))^n=\sum_{i=0}^{b^n-1}1\cdot e^{s_b(i)x}.$$

Thus, for $m>1$, we have

$$\frac{d^m}{dx^m}(g(x))^n=\sum_{i=0}^{b^n-1}(s_b(i))^m e^{s_b(i)x},$$

and so we have that

$$\frac{d^m}{dx^m}(g(0))^n=\sum_{i=0}^{b^n-1}(s_b(i))^m.$$

Now we need Faá di Bruno's formula [8]. This formula states that if $f(x)$ and $g(x)$ are functions for which all the necessary derivatives are defined and m is a positive integer, then

$$\frac{d^m}{dx^m}f(g(x))=\sum_{n_1+2n_2+\cdots+mn_m=m}\frac{m!}{n_1!\cdots n_m!}\left(\frac{d^{n_1+\cdots+n_m}}{dx^{n_1+\cdots+n_m}}f\right)(g(x))$$
$$\cdot\left(\frac{\frac{d}{dx}g(x)}{1!}\right)^{n_1}\cdots\left(\frac{\frac{d^m}{dx^m}g(x)}{m!}\right)^{n_m},$$

where $n_1,n_2,\ldots,n_m$ are nonnegative integers.

It follows that

$$\begin{aligned}\frac{d^m}{dx^m}(g(x))^n = \sum_{n_1+2n_2+\cdots+mn_m=m} & n^{\underline{n_1+n_2+\cdots+n_m}} g(x)^{n-n_1-n_2-\cdots-n_m} \\ & \cdot \frac{m!}{(1!)^{n_1}n_1!(2!)^{n_2}n_2!\cdots(m!)^{n_m}n_m!}(g^{(1)}(x))^{n_1}(g^{(2)}(x))^{n_2}\cdots(g^{(m)}(x))^{n_m},\end{aligned}$$

where m is a positive integer and $n_1, n_2, \ldots, n_m$ are nonnegative integers.

Thus,

$$\begin{aligned}\frac{d^m}{dx^m}(g(0))^n = \sum_{n_1+2n_2+\cdots+mn_m=m} & n^{\underline{n_1+n_2+\cdots+n_m}} g(0)^{n-n_1-n_2-\cdots-n_m} \\ & \cdot \frac{m!}{(1!)^{n_1}n_1!(2!)^{n_2}n_2!\cdots(m!)^{n_m}n_m!}(g^{(1)}(0))^{n_1}(g^{(2)}(0))^{n_2}\cdots(g^{(m)}(0))^{n_m}.\end{aligned}$$

Equating the two expressions for

$$\frac{d^m}{dx^m}(g(0))^n$$

and simplifying gives the following theorem.

Theorem 2. Let n and m be positive integers and $n_1, n_2, \ldots, n_m$ be nonnegative integers. Then

$$\begin{aligned}\frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i))^m = \sum_{n_1+2n_2+\cdots+mn_m=m} & \frac{m!}{(1!)^{n_1}n_1!(2!)^{n_2}n_2!\cdots(m!)^{n_m}n_m!} \\ & \cdot (g^{(1)}(0)/b)^{n_1}(g^{(2)}(0)/b)^{n_2}\cdots(g^{(m)}(0)/b)^{n_m} n^{\underline{n_1+n_2+\cdots+n_m}},\end{aligned}$$

where

$$g^{(i)}(0) = 0^i + 1^i + \cdots + (b-1)^i.$$

It might be noted that in [4], formulas for the sums

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1}(s_{10}(i))^m$$

were given for $m = 0, 1, \ldots, 8$. Using the formulas we just derived, we have the new formula for $m = 9$, that is,

$$\begin{aligned}\frac{1}{10^n}\sum_{i=0}^{10^n-1}(s_{10}(i))^9 &= \frac{387420489}{512}\cdot n^9 + \frac{1420541793}{128}\cdot n^8\\ &+ \frac{12153524229}{256}\cdot n^7 + \frac{7215728751}{160}\cdot n^6\\ &- \frac{30325460319}{512}\cdot n^5 - \frac{2286016425}{128}\cdot n^4\\ &+ \frac{30058716303}{640}\cdot n^3 - \frac{2699999973}{160}\cdot n^2.\end{aligned}$$

3. Third Sum. We next try to tackle the sum

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1} s_b(i)\cdot L_b(i).$$

The base 10 result is

$$\frac{1}{10^n}\sum_{i=0}^{10^n-1} s_{10}(i)\cdot L_{10}(i) = \frac{9}{4}n^2 + \frac{5}{4}n.$$

From the previous two sections, we have established the formulas

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(s_b(i))^2 = \frac{b^2-2b+1}{4}n^2 + \frac{b^2-1}{12}n$$

and

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1}(L_b(i))^2 = \left(\frac{\lfloor b/2\rfloor}{b}\right)^2 n^2 + \left(\left(\frac{\lfloor b/2\rfloor}{b}\right) - \left(\frac{\lfloor b/2\rfloor}{b}\right)^2\right)n.$$

Now consider the function

$$h(x) = (1 + e^x + e^{2x} + \cdots + e^{(\lceil b/2\rceil-1)x} + e^{(\lceil b/2\rceil+1)x} + \cdots + e^{bx})^n$$

The motivation for this function comes from the fact that in the base b representation of $i = i_n\cdots i_2 i_1$, the jth digit of i, i_j, contributes either i_j or i_j+1, depending upon whether or not the i_jth digit is small or large, respectively. That is, the $h(x)$ function considers both the digital sum and the number of large digits, compared to the $g(x)$ function where we were only concerned with the digital sum. Expanding the product, we see that there is a 1-1 correspondence between the numbers $0 \le i \le b^n - 1$ and the b^n terms $1 \cdot e^{(s_b(i)+L_b(i))x}$. Therefore,

$$\begin{aligned} h(x) &= (1 + e^x + e^{2x} + \cdots + e^{(\lceil b/2\rceil - 1)x} + e^{(\lceil b/2\rceil + 1)x} + \cdots + e^{bx})^n \\ &= \sum_{i=0}^{b^n-1} 1 \cdot e^{(s_b(i)+L_b(i))x}. \end{aligned}$$

Thus,

$$h''(x) = \sum_{i=0}^{b^n-1} (s_b(i) + L_b(i))^2 e^{(s_b(i)+L_b(i))x},$$

and so we have that

$$h''(0) = \sum_{i=0}^{b^n-1} (s_b(i) + L_b(i))^2.$$

Computing $h''(0)$ and dividing by b^n, we obtain

$$\begin{aligned} &\frac{1}{b^n} \sum_{i=0}^{b^n-1} (s_b(i) + L_b(i))^2 \\ &= n(n-1)b^{-2} \cdot \left(\frac{b(b+1)}{2} - \left\lceil \frac{b}{2} \right\rceil \right)^2 + nb^{-1} \cdot \left(\frac{b(b+1)(2b+1)}{6} - \left\lceil \frac{b}{2} \right\rceil^2 \right) \\ &= \left(\frac{b^2 + b - 2\lceil b/2\rceil}{2b} \right)^2 n^2 \\ &+ \left(\left(\frac{2b^3 + 3b^2 + b - 6\lceil b/2\rceil^2}{6b} \right) - \left(\frac{b^2 + b - 2\lceil b/2\rceil}{2b} \right)^2 \right) n. \end{aligned}$$

But,

$$\begin{aligned} &\frac{1}{b^n} \sum_{i=0}^{b^n-1} s_b(i) \cdot L_b(i) \\ &= \frac{1}{b^n} \sum_{i=0}^{b^n-1} \frac{(s_b(i) + L_b(i))^2 - (s_b(i))^2 - (L_b(i))^2}{2} \\ &= \frac{1}{2} \left(\frac{1}{b^n} \sum_{i=0}^{b^n-1} (s_b(i) + L_b(i))^2 - \frac{1}{b^n} \sum_{i=0}^{b^n-1} (s_b(i))^2 - \frac{1}{b^n} \sum_{i=0}^{b^n-1} (L_b(i))^2 \right). \end{aligned}$$

Substituting our three formulas in the above expression, we have

$$\begin{aligned}
&\frac{1}{b^n}\sum_{i=0}^{b^n-1} s_b(i)\cdot L_b(i)\\
&=\frac{1}{2}\left(\frac{b^2+b-2\lceil b/2\rceil}{2b}\right)^2 n^2\\
&+\frac{1}{2}\left(\left(\frac{2b^3+3b^2+b-6\lceil b/2\rceil^2}{6b}\right)-\left(\frac{b^2+b-2\lceil b/2\rceil}{2b}\right)^2\right)n\\
&-\frac{1}{2}\left(\frac{b^2-2b+1}{4}n^2+\frac{b^2-1}{12}n\right)\\
&-\frac{1}{2}\left(\left(\frac{\lfloor b/2\rfloor}{b}\right)^2 n^2+\left(\left(\frac{\lfloor b/2\rfloor}{b}\right)-\left(\frac{\lfloor b/2\rfloor}{b}\right)^2\right)n\right).
\end{aligned}$$

Collecting like terms, we have the following theorem.

Theorem 3. Let n be a positive integer. Then

$$\begin{aligned}
&\frac{1}{b^n}\sum_{i=0}^{b^n-1} s_b(i)\cdot L_b(i)\\
&=\frac{1}{2}\left(\left(\frac{b^2+b-2\lceil b/2\rceil}{2b}\right)^2-\frac{b^2-2b+1}{4}-\left(\frac{\lfloor b/2\rfloor}{b}\right)^2\right)n^2\\
&+\frac{1}{2}\left(\left(\frac{2b^3+3b^2+b-6\lceil b/2\rceil^2}{6b}\right)-\left(\frac{b^2+b-2\lceil b/2\rceil}{2b}\right)^2\right.\\
&\left.-\frac{b^2-1}{12}-\left(\left(\frac{\lfloor b/2\rfloor}{b}\right)-\left(\frac{\lfloor b/2\rfloor}{b}\right)^2\right)\right)n.
\end{aligned}$$

Furthermore, we have the following corollary.

Corollary. Let n be a positive integer and b be a positive even integer. Then

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1} s_b(i)\cdot L_b(i)=\frac{b-1}{4}n^2+\frac{b}{8}n.$$

4. Fourth Sum. We next determine the sum

$$\sum_{i=1}^{\infty}\frac{C_b(a;\ a^i)}{(s_b(a))^i},$$

where

$$C_b(x;\ y)$$

denotes the sum of the carries when the positive integer x is multiplied by y, using the normal multiplication algorithm in base b arithmetic.

Noting that

$$L_{10}(2^i) = C_{10}(2;\ 2^i),$$

this sum is a generalization of the sum

$$\sum_{i=1}^{\infty} \frac{L_{10}(2^i)}{2^i}$$

which was a problem considered in [6].

To compute this sum, we need the following lemma.

Lemma 1. Let d be a digit in base b and y be any positive integer. Then

$$C_b(d;\ y) = \frac{1}{b-1}(d \cdot s_b(y) - s_b(dy)).$$

Proof. The proof of Lemma 1 relies on Legendre's theorem

$$s_b(n) = n - (b-1)\sum_{t\geq 1} \left\lfloor \frac{n}{b^t} \right\rfloor,$$

where n is a positive integer. Legendre's theorem and its proof can be found in [5].

To prove Lemma 1, we note that

$$s_b(y) = y - (b-1)\sum_{t\geq 1} \left\lfloor \frac{y}{b^t} \right\rfloor$$

and

$$s_b(dy) = dy - (b-1)\sum_{t\geq 1} \left\lfloor \frac{dy}{b^t} \right\rfloor.$$

Multiplying the first equality by d and subtracting the second equality from the first yields

$$d \cdot s_b(y) - s_b(dy) = (b-1)\sum_{t\geq 1} \left(\left\lfloor \frac{dy}{b^t} \right\rfloor - d \left\lfloor \frac{y}{b^t} \right\rfloor \right).$$

Dividing by $b-1$ and observing that the sum is $C(d;\ y)$ gives us the result.

Armed with Lemma 1, we have the next lemma.

Lemma 2. Let $s_b(n)$ denote the base b digital sum of the positive integer n and $C_b(a;\ a^i)$ denote the base b carries in the normal multiplication algorithm of multiplying a and a^i. Let x and y be positive integers. Then

$$s_b(x \cdot y) = s_b(x) \cdot s_b(y) - (b-1)C_b(x;\ y).$$

Proof. Consider

$$x = \sum_{i=0}^{n} x_i b^i,$$

the base b representation of x. Then counting the top carries from the multiplication using Lemma 1 and counting the bottom carries from the addition, we have

$$\begin{aligned}
C_b(x;\ y) &= \frac{1}{b-1}\sum_{i=0}^{n}(x_i s_b(y) - s_b(x_i y)) \\
&+ \sum_{t\geq 1}\left(\left\lfloor \frac{\sum_{i=0}^{n} x_i b^i y}{b^t}\right\rfloor - \sum_{i=0}^{n}\left\lfloor \frac{x_i b^i y}{b^t}\right\rfloor\right) \\
&= \frac{1}{b-1}s_b(x)s_b(y) - \frac{1}{b-1}\sum_{i=0}^{n} s_b(x_i y) \\
&+ \sum_{t\geq 1}\left\lfloor \frac{xy}{b^t}\right\rfloor - \sum_{i=0}^{n}\sum_{t\geq 1}\left\lfloor \frac{x_i b^i y}{b^t}\right\rfloor \\
&= \frac{1}{b-1}s_b(x)s_b(y) - \frac{1}{b-1}\sum_{i=0}^{n} s_b(x_i y) \\
&+ \frac{1}{b-1}(xy - s_b(xy)) \\
&- \sum_{i=0}^{n}\frac{1}{b-1}(x_i b^i y - s_b(x_i b^i y)) \\
&= \frac{1}{b-1}(s_b(x)s_b(y) - s_b(xy)).
\end{aligned}$$

Next, applying Lemma 2, we obtain

$$s_b(a^{i+1}) = s_b(a) \cdot s_b(a^i) - (b-1)C_b(a;\ a^i).$$

Thus, if n is a positive integer,

$$\begin{aligned}\sum_{i=1}^{n} \frac{C_b(a;\ a^i)}{s_b(a)^i} &= \frac{1}{b-1}\sum_{i=1}^{n}\left(\frac{s_b(a^i)}{(s_b(a))^{i-1}} - \frac{s_b(a^{i+1})}{(s_b(a))^i}\right)\\ &= \frac{1}{b-1}s_b(a) - \frac{1}{b-1}\frac{s_b(a^{n+1})}{(s_b(a))^n}.\end{aligned}$$

Therefore, we have the following theorem.

Theorem 4. Let $s_b(n)$ denote the base b digital sum of the positive integer n and $C_b(a;\ a^i)$ denote the base b carries in the normal multiplication algorithm of multiplying a and a^i. Then

$$\sum_{i=1}^{\infty} \frac{C_b(a;\ a^i)}{(s_b(a))^i} = \frac{s_b(a)}{b-1}.$$

To illustrate this theorem, if $b = 3$ and $a = 14$, then

$$\sum_{i=1}^{\infty} \frac{C_3(14;\ 14^i)}{4^i} = 2.$$

That is, if we count the carries in multiplying $14 = 112_3$ by powers of 14, using the usual base 3 multiplication algorithm, and divide by the appropriate power of 4, the result is 2. In fact, the infinite series begins with

$$\frac{5}{4} + \frac{7}{16} + \frac{14}{64} + \frac{18}{256} + \cdots .$$

5. Questions. Some open questions remain. Can a formula be found for

$$\frac{1}{b^n}\sum_{i=0}^{b^n-1} (s_b(i))^{n_1} \cdot (L_b(i))^{n_2},$$

where n, n_1, and n_2 are positive integers? Can a formula be found for

$$\frac{1}{b^n}\sum_{i=1}^{b^n-1} \frac{1}{s_b(i)}?$$

Also, can a formula be found for

$$\frac{1}{b_1^n}\sum_{i=0}^{b_1^n-1} s_{b_1}(i) \cdot s_{b_2}(i),$$

where $b_1 = b_2^m$? What about a formula for

$$\frac{1}{b^n} \sum_{i=0}^{b^n-1} s_b(s_b(i))?$$

Finally, find the sum

$$\sum_{i=1}^{\infty} \frac{s_b(a^i)}{a^i}.$$

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