ON THE SET OF POSITIVE INTEGERS WHICH ARE RELATIVELY PRIME TO THEIR DIGITAL SUM AND ITS COMPLEMENT

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0. Abstract. We will consider the set $\{k \mid (k, s(k)) = 1\}$ and its complement. Here, $s(k)$ denotes the base 10 digital sum of k and (m, n) denotes the greatest common divisor of m and n. First, we will give an upper bound for the natural density of $\{k \mid (k, s(k)) = 1\}$, if it exists. Next, we will show that for each m, there exist m consecutive integers $N + 1, \ldots, N + m$ such that, for each $k \in \{N + 1, \ldots, N + m\}$, $(k, s(k)) \neq 1$.

1. Upper Bound for Natural Density. The natural density of certain sets of positive integers has intrigued mathematicians for centuries. If A denotes a set of positive integers and $A(x)$ denotes the number of positive integers in A not exceeding x, then the natural density of the set A is

$$\lim_{x \to \infty} \frac{A(x)}{x},$$

if the limit exists. For example, the set of primes has natural density of 0. In fact, if $\pi(x)$ denotes the number of prime numbers not exceeding x, then

$$\lim_{x \to \infty} \frac{\pi(x)}{x / \log x} = 1.$$

Here, $\log x$ is the natural logarithm of x. A proof of this result can be found in [1]. In [2], the natural density of the squarefree numbers is shown to be $6/\pi^2$.

More recently, several sets of positive integers have been defined with respect to the digital sum function. Two of the better known examples are the sets of Niven numbers and Smith numbers. A Niven number, named in honor of Professor Ivan Niven, is a positive integer which is divisible by its digital sum. For example, 21 is a Niven number since 21 is divisible by $2 + 1$. Several articles including [3], [4], and [5] have been written regarding Niven numbers. It was shown in [3] that there are an infinite number of Niven numbers, since every integral power of 10 is a Niven number. In [6], the natural density of the Niven numbers was shown to be 0. The proof involved some statistics and Chebyshev's inequality.

A Smith number is a positive integer such that the sum of its digits is equal to the sum of the sum of the digits of each of its prime factors (taken with multiplicity). For example, 22 is a Smith number since $22 = 2 \cdot 11$ and $s(22) = s(2) + s(11)$. The article which pioneered the study of Smith numbers is [7]. In [8], it was shown that there are infinitely many Smith numbers. However, we have not found an answer to the density question for Smith numbers. In particular, if $S(x)$ denotes the number of Smith numbers not exceeding x, what is

$$\lim_{x\to\infty} \frac{S(x)}{x},$$

if it exists? We conjecture that the natural density of the Smith numbers is 0.

In this paper, we will consider the set

$$A = \{k \mid (k, s(k)) = 1\}.$$

To attack the problem of finding an upper bound for the natural density of A, let

$$\begin{aligned} P_2 &= \{k \mid 2|k \text{ and } 2|s(k)\}, \\ P_3 &= \{k \mid 3|k \text{ and } 3|s(k)\}, \text{ and} \\ P_5 &= \{k \mid 5|k \text{ and } 5|s(k)\}. \end{aligned}$$

Using the notation $d(B)$ to denote the natural density of the set B, we have that $d(P_2) = 1/4$, $d(P_3) = 1/3$, and $d(P_5) = 1/25$. Since A and $P_2 \cup P_3 \cup P_5$ are disjoint, we have that

$$d(A) \leq 1 - d(P_2 \cup P_3 \cup P_5),$$

if $d(A)$ exists. Using the inclusion-exclusion principle [9],

$$\begin{aligned} d(P_2 \cup P_3 \cup P_5) &= d(P_2) + d(P_3) + d(P_5) - d(P_2 \cap P_3) \\ &\quad - d(P_2 \cap P_5) - d(P_3 \cap P_5) + d(P_2 \cap P_3 \cap P_5) \\ &= \frac{1}{4} + \frac{1}{3} + \frac{1}{25} - \frac{1}{4 \cdot 3} - \frac{1}{4 \cdot 25} - \frac{1}{3 \cdot 25} + \frac{1}{4 \cdot 3 \cdot 25} \\ &= \frac{13}{25}. \end{aligned}$$

Thus $d(A)$, if it exists, is less than or equal to $12/25$.

The above argument can be repeated for any set of primes. Let n be a positive integer and let p_i denote the ith prime. Now choose the collection of primes $p_1 = 2, p_2 = 3, p_3 = 5, \ldots, p_n$. Defining C_n and simplifying, we have that

$$\begin{aligned}
C_n &= d(P_{p_1} \cup \cdots \cup P_{p_n}) = \sum_{1 \le i \le n} d(P_{p_i}) \\
&- \sum_{1 \le i < j \le n} d(P_{p_i} \cap P_{p_j}) + \sum_{1 \le i < j < k \le n} d(P_{p_i} \cap P_{p_j} \cap P_{p_k}) \\
&+ \cdots + (-1)^{n-1} d(P_{p_1} \cap \cdots \cap P_{p_n}) \\
&= \frac{1}{4} + \frac{1}{3} + \frac{1}{25} + \cdots + \frac{1}{p_n^2} - \frac{1}{4 \cdot 3} - \frac{1}{4 \cdot 25} - \frac{1}{3 \cdot 25} - \cdots - \frac{1}{p_{n-1}^2 \cdot p_n^2} \\
&+ \frac{1}{4 \cdot 3 \cdot 25} + \cdots + (-1)^{n-1} \frac{1}{4 \cdot 3 \cdot 25 \cdot \cdots \cdot p_n^2}
\end{aligned}$$

Therefore, an upper bound for $d(A)$, if it exists, is $1 - C_n$. Furthermore, if we could find

$$C = \lim_{n \to \infty} C_n,$$

then an upper bound for $d(A)$, if it exists, is $1 - C$.

To determine C we let

$$\begin{aligned}
x = &\frac{1}{2^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \cdots \\
&- \frac{1}{2^2 \cdot 5^2} - \frac{1}{2^2 \cdot 7^2} - \frac{1}{5^2 \cdot 7^2} - \cdots \\
&+ \frac{1}{2^2 \cdot 5^2 \cdot 7^2} + \cdots
\end{aligned}$$

and

$$\begin{aligned}
K = &\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \cdots \\
&- \frac{1}{2^2 \cdot 3^2} - \frac{1}{2^2 \cdot 5^2} - \frac{1}{3^2 \cdot 5^2} - \cdots \\
&+ \frac{1}{2^2 \cdot 3^2 \cdot 5^2} + \cdots .
\end{aligned}$$

Then,

$$x + \frac{1}{3^2} - \frac{1}{3^2 \cdot 2^2} - \frac{1}{3^2 \cdot 5^2} - \cdots + \frac{1}{3^2 \cdot 2^2 \cdot 5^2} + \cdots = K.$$

Therefore,

$$x + \frac{1}{3^2} - \frac{1}{3^2} x = K$$

and so,

$$x = \frac{9}{8}K - \frac{1}{8}.$$

Now,

$$\begin{aligned} C &= x + \frac{1}{3} - \frac{1}{3 \cdot 2^2} - \frac{1}{3 \cdot 5^2} - \cdots + \frac{1}{3 \cdot 2^2 \cdot 5^2} + \cdots \\ &= x + \frac{1}{3} - \frac{1}{3}x = \frac{2}{3}x + \frac{1}{3} \\ &= \frac{2}{3}\left(\frac{9}{8}K - \frac{1}{8}\right) + \frac{1}{3} = \frac{3}{4}K + \frac{1}{4}. \end{aligned}$$

But from [2], the density of the "squarefull" numbers is $K = 1 - 6/\pi^2$. Hence $C = 1 - 9/2\pi^2$, and so $1 - C = 9/2\pi^2$. A decimal approximation to $9/2\pi^2$ is 0.4559454.

2. Consecutiveness in the Complement Set. In attempting to learn more about

$$A = \{k \mid (k, s(k)) = 1\},$$

we wondered about the length of the longest sequence of consecutive integers in A. From past experience, we found such questions to be quite fruitful. For example in [10], we found that the largest number of consecutive Niven numbers which exist is 20 and that there are an infinite number of groups of 20 consecutive Niven numbers. Therefore, we approached this problem with some anticipation. However, it soon became clear that there can be no more than 2 consecutive integers in A, since $3|k$ if and only if $3|s(k)$. Also for $k \geq 1$, the consecutive integers 10^k and $10^k + 1$ demonstrate that there are an infinite number of groups of 2 consecutive integers such that $(k, s(k)) = 1$.

Since the above problem turned out to be trivial, we decided to examine the complement of A and ask about the length of the longest sequence of consecutive integers in the complement of A.

After some creative searching, we found the following lists of 12 consecutive integers which are in the complement of A.

k	$s(k)$	$(k, s(k))$
$180090_{2j}20$	20	20
$180090_{2j}21$	21	3
$180090_{2j}22$	22	2
$180090_{2j}23$	23	23
$180090_{2j}24$	24	24
$180090_{2j}25$	25	25
$180090_{2j}26$	26	2
$180090_{2j}27$	27	27
$180090_{2j}28$	28	4
$180090_{2j}29$	29	29
$180090_{2j}30$	21	3
$180090_{2j}31$	22	11

Here, $j \geq 0$. In addition, the following list of 29 consecutive integers is in the complement of A.

k	$s(k)$	$(k, s(k))$
66166892131839499000000017947066278894975530189	216	9
66166892131839499000000017947066278894975530190	208	2
66166892131839499000000017947066278894975530191	209	19
66166892131839499000000017947066278894975530192	210	6
66166892131839499000000017947066278894975530193	211	211
66166892131839499000000017947066278894975530194	212	2
66166892131839499000000017947066278894975530195	213	3
66166892131839499000000017947066278894975530196	214	2
66166892131839499000000017947066278894975530197	215	43
66166892131839499000000017947066278894975530198	216	18
66166892131839499000000017947066278894975530199	217	31
66166892131839499000000017947066278894975530200	200	200
66166892131839499000000017947066278894975530201	201	3
66166892131839499000000017947066278894975530202	202	2
66166892131839499000000017947066278894975530203	203	7
66166892131839499000000017947066278894975530204	204	12
66166892131839499000000017947066278894975530205	205	5
66166892131839499000000017947066278894975530206	206	2
66166892131839499000000017947066278894975530207	207	9
66166892131839499000000017947066278894975530208	208	16
66166892131839499000000017947066278894975530209	209	11
66166892131839499000000017947066278894975530210	201	3
66166892131839499000000017947066278894975530211	202	101
66166892131839499000000017947066278894975530212	203	29
66166892131839499000000017947066278894975530213	204	3
66166892131839499000000017947066278894975530214	205	41
66166892131839499000000017947066278894975530215	206	103
66166892131839499000000017947066278894975530216	207	9
66166892131839499000000017947066278894975530217	208	13

Motivated by these two examples, we were led to the following lemmas and a consecutiveness theorem. Lemma 1 is a rather famous result from additive number theory.

Lemma 1. Let a and b be relatively prime positive integers. Then

$$ax + by = n$$

has a solution in nonnegative integers x and y, if n is large enough.
For more information about Lemma 1 and related topics, see [11].

Lemmas 2 and 3 were stated and proved in [12]. We will include their proofs due to the fact that their reference is not well-known.

Lemma 2. Let $(b, 30) = 1$. Then there exist x and y, both multiples of b, such that $(s(x), s(y)) = 1$.

Proof. Let $\exp_b(10)$ denote the order of 10 modulo b under multiplication. The proof divides into two cases. The first case is when the last digit of b is 1, 3, or 7 and the second case is when the last digit of b is a 9. Note that b cannot end in an even digit or a 5 since $(b, 30) = 1$.

Case 1. Suppose the ending digits of b are $z0_t d$ where $z \neq 0$ and $d = 1, 3$, or 7. Here, a subscript on a number or digit denotes that number or digit concatenated with itself as many times as specified by the subscript.

Subcase a. $t = 0$. Let

$$x = (d+1) \cdot 10^{\exp_b(10)} + b - (d+1) \text{ and } y = b.$$

Both x and y are multiples of b. In addition, $s(x) = s(b) + 9$ and $s(y) = s(b)$. Since $(b, 30) = 1$, $(s(x), s(y)) = 1$.

Subcase b. $t > 0$. Let

$$x = 10^{t+\exp_b(10)} + b - 10^t \text{ and } y = b.$$

Both x and y are multiples of b. In addition, $s(x) = s(b) + 9$ and $s(y) = s(b)$ so $(s(x), s(y)) = 1$.

Case 2. Suppose the ending digits of b are $z9_t$ where $t \geq 1$ and $z \neq 9$. Let

$$x = b \cdot 10^{\exp_b(10)} + b \text{ and } y = (b - 10^t) \cdot 10^{\exp_b(10)-1} + b + 10^{t-1}.$$

Both x and y are multiples of b. In addition, $s(x) = 2s(b)$ and $s(y) = 2s(b) - 9$ so $(s(x), s(y)) = 1$.

Now we are ready for Lemma 3.

Lemma 3. Let $(b, 3) = 1$. Then

$$s(z) = n$$

has a solution which is a multiple of b, if n is large enough.

Proof. We assume, without loss of generality, that $(b, 10) = 1$. To see this, suppose that $b = 2^e 5^f b_1$, where $(b_1, 10) = 1$. Also suppose that for b_1 we can find an N such that if $n \geq N$, then there exists a multiple of b_1, z_1, such that $s(z_1) = n$. Now let $g = \max(e, f)$ and $z = 10^g \cdot z_1$. Then since z_1 is a multiple of b_1, z is a multiple of b. In addition, $s(z) = n$.

Since $(b, 3) = 1$, $(b, 30) = 1$. Thus, by Lemma 2, there exist two multiples of b, x and y, such that $(s(x), s(y)) = 1$. Next by the Lemma 1,

$$As(x) + Bs(y) = n$$

has a solution in nonnegative integers A and B, if n is large enough. Now if we let $z = x_A y_B$, z is a multiple of b and $s(z) = n$.

Finally, the theorem below turned out to be quite surprising.

Theorem. There exist arbitrarily long sequences of consecutive positive integers k such that $(k, s(k)) \neq 1$.

Proof. Let m be a positive integer and $p_0, p_1, \ldots, p_{10^m-1}$ be distinct primes larger than 10^m.

We begin by solving the first system of congruences

$$\begin{aligned}
r \cdot 10^m &+ 0 \equiv 0 \pmod{p_0} \\
r \cdot 10^m &+ 1 \equiv 0 \pmod{p_1} \\
r \cdot 10^m &+ 2 \equiv 0 \pmod{p_2} \\
&\vdots \\
r \cdot 10^m &+ (10^m - 1) \equiv 0 \pmod{p_{10^m-1}}.
\end{aligned}$$

The general solution to this first system of congruences is $r = a + bk$, where a and b are positive constants and k is an integer [1]. Note that $(b, 3) = 1$.

Next, we will solve a second system of congruences

$$\begin{aligned}
t+s(0) &\equiv 0 \pmod{p_0} \\
t+s(1) &\equiv 0 \pmod{p_1} \\
t+s(2) &\equiv 0 \pmod{p_2} \\
&\vdots \\
t+s(10^m-1) &\equiv 0 \pmod{p_{10^m-1}}.
\end{aligned}$$

The general solution to this second system of congruences is $t = c + dk$, where c and d are positive constants and k is an integer.

Now, pick a particular solution t to the second system of congruences such that $t - s(a)$ is large enough (according to Lemma 3 for b). That is, $t - s(a)$ is large enough so that

$$s(z) = t - s(a)$$

has a solution which is a multiple of b.

Finally, letting

$$r = a + z \cdot 10^{\lfloor \log a \rfloor + 1},$$

we have that

$$\begin{aligned}
s(r) &= s(a + z \cdot 10^{\lfloor \log a \rfloor + 1}) \\
&= s(a) + s(z) = s(a) + t - s(a) = t.
\end{aligned}$$

Due to the manner in which r is constructed, p_i is a factor of both $r \cdot 10^m + i$ and $s(r \cdot 10^m + i)$ for $i = 0, 1, \ldots, 10^m - 1$. Therefore, the 10^m numbers $r \cdot 10^m + 0$, $r \cdot 10^m + 1, \ldots, r \cdot 10^m + (10^m - 1)$ satisfy $(k, s(k)) \neq 1$ and the theorem is proved.

3. Conclusion. The set

$$\{k \mid (k, s(k)) = 1\}$$

and its complement have produced some interesting results. We would very much like to determine the natural density of this set, if it exists? In addition, it might be fruitful to explore the set

$$\{k \mid (k, s_b(k)) = 1\},$$

where $s_b(k)$ denotes the base b digital sum of k. Also, the set

$$\{k \mid (k, s(k)) = g\},$$

where g is a positive integer may prove worthy of study.

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