DIVERGENT RATS SEQUENCES

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1. Introduction. In 1990 John Conway invented a digital game called RATS [1]. RATS is an acronym for Reverse, Add, Then Sort. A game of RATS produces a sequence of positive integers. Each positive integer in the sequence has its digits arranged in nondecreasing order. To play a “game” of RATS we take a positive integer whose digits are arranged in nondecreasing order. Reverse the digits, Add the reversed digits to the number, delete the zero digits in the sum, and Then Sort the remaining digits of the sum in nondecreasing order. The resulting number is the next number in the sequence.

For example, if we begin a game of RATS with 3, assuming base 10, then the RATS sequence is

$$3,\ 6,\ 12,\ 33,\ 66,\ 123,\ 444,\ 888,\ 1677,\ 3489,\ 12333,\ 44556,$$
$$111,\ 222,\ 444,\ 888,\ 1677,\ 3489,\ 12333,\ 44556,\ \ldots$$

which exhibits a cycle of length 8 and least member 111.

In [5], Curt McMullen gave a list of the base 10 RATS cycles he had discovered. Computer searches were done by Curtis Cooper and Robert E. Kennedy [2,3] to find more base 10 cycles. A list of these cycles and the search techniques used can be found in [2,3].

We also have sequences that diverge. The most fundamental one, in base 10, starts with 1.

$$1,\ 2,\ 4,\ 8,\ 16,\ 77,\ 145,\ 668,\ 1345,\ 6677,\ 13444,\ 55778,\ 133345,$$
$$666677,\ 1333444,\ 5567777,\ 12333445,\ 66666677,\ 133333444,\ 556667777,$$
$$1233334444,\ 5566667777,$$
$$\vdots$$

Notice that in each successive number, the number of 3's and 6's both increase by 1. This is the mark of a divergent sequence. This sequence, in particular, is known as Conway's Divergent Sequence.

Due to the size and repetitive nature of the digits in the RATS game, we will use superscripts to denote repeated digits in a number. For example, 11122223344444, will be represented as $1^3 2^4 3^2 4^5$. Using this notation, we can give the closed form of Conway's Divergent Sequence.

Conway's Divergent Sequence. Let $m \geq 2$. Then

$$1\ 2\ 3^m\ 4^4\ ,\ 5^2\ 6^m\ 7^4$$

is a length 2 divergent sequence in base 10. Here, length 2 means that the sequence "comes back to itself" after the second iteration.

This paper will emphasize divergent sequences for bases other than 10. Some preliminary work has already been done for bases 19, 37, and 50 by McMullen [5]. In bases larger than 10, the digits bigger than 10 will be denoted with parentheses around them.

Lemma 1. Let $m \geq 19$. Then

$$1\ 2\ 3^3\ 4^4\ 5^{12}\ 6^m\ 7^{40}\ ,\ 8^2\ 9^2\ (10)^6\ (11)^8\ (12)^{m+5}\ (13)^{38}$$

is a length 2 divergent sequence in base 19.

Lemma 2. Let $m \geq 1257$. Then

$$\begin{aligned}&1\ 2\ 3^3\ 4^4\ 5^{12}\ 6^{16}\ 7^{48}\ 8^{64}\ 9^{192}\\ &\quad (10)^{256}\ (11)^{768}\ (12)^m\ (13)^{2616},\\ &(14)^2\ (15)^2\ (16)^6\ (17)^8\ (18)^{24}\ (19)^{32}\ (20)^{96}\ (21)^{128}\ (22)^{384}\\ &\quad (23)^{512}\ (24)^{m+285}\ (25)^{2502}\end{aligned}$$

is a length 2 divergent sequence in base 37.

Cooper and Gentges [4] found a divergent sequence in base 55.

Lemma 3. Let $m \geq 80099$. Then

$$1\ 2\ 3^3\ 4^4\ 5^{12}\ 6^{16}\ 7^{48}\ 8^{64}\ 9^{192}\ (10)^{256}\ (11)^{768}\ (12)^{1024}\ (13)^{3072}$$
$$(14)^{4096}\ (15)^{12288}\ (16)^{16384}\ (17)^{49152}\ (18)^{m}\ (19)^{167480},$$
$$(20)^{2}\ (21)^{2}\ (22)^{6}\ (23)^{8}\ (24)^{24}\ (25)^{32}\ (26)^{96}\ (27)^{128}\ (28)^{384}\ (29)^{512}$$
$$(30)^{1536}\ (31)^{2048}\ (32)^{6144}\ (33)^{8192}\ (34)^{24576}\ (35)^{32768}\ (36)^{m+18205}\ (37)^{160198}$$

is a length 2 divergent sequence in base 55.

Finally, Cooper and Gentges [4] found a closed form for the family of divergent sequences in bases $18n+1$, $n \geq 1$.

Theorem 4. Let m be a large positive integer and let $18n+1$ be the base, for $n \geq 1$. Then

$$123^3 4^4 5^{12} 6^{16} 7^{48} 8^{64} \cdots (6n)^m (6n+1)^{(23\cdot 64^n - 32)/36},$$
$$(6n+2)^2(6n+3)^2(6n+4)^6(6n+5)^8(6n+6)^{24}(6n+7)^{32}(6n+8)^{96}(6n+9)^{128}$$
$$\cdots (12n)^{m+(5\cdot 64^n+40)/72}(12n+1)^{(22\cdot 64^n-40)/36}$$

is a length 2 divergent sequence.

All of these divergent sequences have been of length 2. This paper will examine divergent RATS sequences of length $t \geq 2$. First we will show other divergent RATS sequences of length 2. Next, we will show explicit divergent RATS sequences of lengths 3, 4, 5, and 6. In addition, we will prove that there are arbitrarily long divergent RATS sequences.

2. Divergent RATS Sequences of Length 2. Divergent sequences consisting of two elements were found for bases 28, 46, and 64. This led to finding a closed form for divergent sequences of length 2 in base $18n+10$ where $n \geq 1$.

Lemma 5. Let $m \geq 191$. Then

$$1\ 2\ 3^3\ 4^4\ 5^{12}\ 6^{16}\ 7^{48}\ 8^{64}\ 9^m (10)^{312},$$

$$(11)^2\ (12)^2\ (13)^6\ (14)^8\ (15)^{24}\ (16)^{32}\ (17)^{96}\ (18)^{m-35}\ (19)^{326}.$$

is a length 2 divergent sequence in base 28.

The interested reader can obtain the proof from the authors.

By finding similar patterns for bases 46 and 64, we were led to the following closed form for a length 2 divergent RATS sequence in base $18n+10$ where $n \in \mathbb{Z}^+$.

Theorem 6. Let m be a large positive integer. Then

$$1\ 2\ 3^3\ 4^4\ 5^{12}\ 6^{16}\ \cdots\ (6n+1)^{(3\cdot 64^n)/4}\ (6n+2)^{64^n}\ (6n+3)^m\ (6n+4)^{(44\cdot 64^n-8)/9},$$

$$(6n+5)^2\ (6n+6)^2\ (6n+7)^6\ (6n+8)^8\ (6n+9)^{24}\ (6n+10)^{32}\ \cdots$$

$$\cdots\ (12n+6)^{m-(5\cdot 64^n-5)/9}\ (12n+7)^{(46\cdot 64^n-10)/9}$$

is a length 2 divergent sequence in base $18n+10$, $n \geq 1$.

Again, the interested reader can obtain the proof from the authors.

3. Divergent RATS Sequences of Length 3, 4, 5, and 6. In this section, we set out to find divergent RATS sequences of length 3 and longer in different bases. McMullen [5] had discovered the following divergent sequence in base 50. The interested reader can obtain the proof from the authors.

Lemma 7. Let $m \geq 55$. Then

$$1\ 3\ 4^7\ 6^8\ 7^m\ 8^{40},$$

$$9^2\ (11)^2\ (12)^{14}\ (14)^{m-7}\ (15)^{46},$$

$$(24)^4\ (26)^4\ (27)^{28}\ (28)^{m-35}\ (29)^{56}$$

is a length 3 divergent sequence in base 50.

Using the same proof technique as above, the following two lemmas can be proved.

Lemma 8. Let $m \geq 2591$. Then

$$1\ 3\ 4^7\ 6^8\ 7^{56}\ 9^{64}\ (10)^{448}\ (12)^{512}\ (13)^{3584}\ (14)^m\ (15)^{7272},$$
$$(16)^2\ (18)^2\ (19)^{14}\ (21)^{16}\ (22)^{112}\ (24)^{128}\ (25)^{896}\ (27)^{1024}\ (28)^{m+4577}\ (29)^{5182},$$
$$(45)^4\ (47)^4\ (48)^{28}\ (50)^{32}\ (51)^{224}\ (53)^{256}\ (54)^{1792}\ (56)^{m+3637}\ (57)^{5976}$$

is a length 3 divergent sequence in base 99.

Lemma 9. Let $m \geq 797131$. Then

$$1\ 3\ 4^7\ 6^8\ 7^{56}\ 9^{64}\ (10)^{448}\ (12)^{512}$$
$$(13)^{3584}\ (15)^{4096}\ (16)^{28672}\ (18)^{32768}\ (19)^{229376}\ (21)^m\ (22)^{765032},$$
$$(23)^2\ (25)^2\ (26)^{14}\ (28)^{16}\ (29)^{112}\ (31)^{128}\ (32)^{896}\ (34)^{1024}$$
$$(35)^{7168}\ (37)^{8192}\ (38)^{57344}\ (40)^{65536}\ (41)^{458752}\ (42)^{m-465439}\ (43)^{930878},$$
$$(66)^4\ (68)^4\ (69)^{28}\ (71)^{32}\ (72)^{224}\ (74)^{256}\ (75)^{1792}\ (77)^{2048}$$
$$(78)^{14336}\ (80)^{16384}\ (81)^{114688}\ (83)^{131072}\ (84)^{m+120373}\ (85)^{663384},$$

is a length 3 divergent sequence in base 148.

Searching next for a base with a length 4 divergent sequence, we found the following number in base 226. The proof of this lemma is similar to the previous proofs.

Lemma 10. Let $m \geq 8683$. Then

$$1\ 4\ 5^{15}\ 8^{16}\ 9^{240}\ (12)^{256}\ (13)^{3840}\ (15)^m\ (16)^{10176},$$
$$(17)^2\ (20)^2\ (21)^{30}\ (24)^{32}\ (25)^{480}\ (28)^{512}\ (29)^{7680}\ (30)^{m-5807}\ (31)^{11614},$$
$$(48)^4\ (51)^4\ (52)^{60}\ (55)^{64}\ (56)^{960}\ (59)^{1024}\ (60)^{m+6677}\ (61)^{5752},$$
$$(109)^8\ (112)^8\ (113)^{120}\ (116)^{128}\ (117)^{1920}\ (120)^{m+5089}\ (121)^{7272}$$

is a length 4 divergent sequence in base 226.

Searching next for a base with a length 5 divergent sequence we found the following number in base 962.

Lemma 11. Let $m \geq 1040187391$. Then

$$1\ 5\ 6^{31}\ (10)^{32}\ (11)^{992}\ (15)^{1024}\ (16)^{31744}\ (20)^{32768}$$
$$(21)^{1015808}\ (25)^{1048576}\ (26)^{32505856}\ (30)^{33554432}\ (31)^{m}\ (32)^{213407584},$$
$$(33)^{2}\ (37)^{2}\ (38)^{62}\ (42)^{64}\ (43)^{1984}\ (47)^{2048}\ (48)^{63488}$$
$$(53)^{65536}\ (54)^{2031616}\ (58)^{65011712}\ (62)^{m-780107455}\ (63)^{290432638},$$
$$(96)^{4}\ (100)^{4}\ (101)^{124}\ (105)^{128}\ (106)^{3968}\ (110)^{4096}\ (111)^{126976}\ (115)^{131072}$$
$$(116)^{4063232}\ (120)^{4194304}\ (121)^{130023424}\ (124)^{m-299266427}\ (125)^{442317944},$$
$$(221)^{8}\ (225)^{8}\ (226)^{248}\ (230)^{256}\ (231)^{7936}\ (235)^{8192}\ (236)^{253952}\ (240)^{262144}$$
$$(241)^{8126464}\ (245)^{8388608}\ (246)^{260046848}\ (248)^{m-603037029}\ (249)^{607541224},$$
$$(470)^{16}\ (474)^{16}\ (475)^{496}\ (479)^{512}\ (480)^{15872}\ (484)^{16384}\ (485)^{507904}\ (489)^{524288}$$
$$(490)^{16252928}\ (494)^{16777216}\ (495)^{520093696}$$
$$(496)^{566516434}\ (497)^{m-933483599}\ (498)^{660893120}$$

is a length 5 divergent sequence in base 962.

Lemma 12. Let m be a large positive integer. Then

$$1\ 6\ 7^{63}\ (12)^{64}\ (13)^{4032}\ (18)^{4096}\ (19)^{258048}\ (24)^{262144}$$
$$(25)^{16515072}\ (30)^{16777216}\ (31)^{1056964608}\ (36)^{1073741824}\ (37)^{67645734912}$$
$$(42)^{68719476736}\ (43)^{4329327034368}\ (48)^{4398046511104}\ (49)^{277076930199552}$$
$$(54)^{281474976710656}\ (55)^{17732923532771328}\ (60)^{18014398509481984}$$
$$(61)^{1134907106097364992}\ (63)^{m}(64)^{2472288970952097408}$$

is one of 6 numbers in a length 6 divergent sequence in base 3970.

4. Arbitrarily Long Divergent RATS Sequences. The arbitrarily long divergent RATS sequences will follow the patterns of the divergent RATS sequences we have seen in the previous chapters. We will come as close to explicitly constructing divergent RATS sequences as possible. We will state the base of operation, the form of the smallest element in the divergent RATS sequence, and the exponents (repetition factors) of each of the elements in the smallest element in the divergent sequence. The only part which will be left to the imagination is one of the exponents, which is a component solution to a particular linear system.

The use of primes and pseudoprimes in the following came about by doing extensive trials on numerous different lengths and noticing a pattern for all lengths that were prime or pseudoprime. This pattern is best described as follows. Divergent sequences of length t, where t is prime or pseudoprime, have three consecutive integers at the end of the first term of the sequence. After one iteration of RATS is complete, there are only two consecutive integers at the end and the prior number is $t-1$ smaller than the first consecutive integer. This "gap" shrinks by one at each successive iteration until in the $(t+1)$st iteration, the gap closes and we return to three consecutive integers at the end. Then this pattern begins again. Other patterns may exist for lengths that are not prime or pseudoprime.

Let t be a prime or a pseudoprime in base 2. A pseudoprime in base 2 [7] is a composite number n such that

$$2^n \equiv 2 \pmod{n}.$$

The smallest pseudoprime in base 2 (psp) is 341. Since t is either a prime or a psp, it follows that

$$2^t \equiv 2 \pmod{t}.$$

The preceding considerations lead to Theorem 13.

<u>Theorem 13</u>. Let m be a large positive integer and t be a prime or a pseudoprime in base 2. Let

$$\begin{aligned}
a_1 &= 1 + 1 + (2^t - 1) + 2^t + \cdots + (2^{2^t-2-t} - 2^{2^t-2-2t}) + (2^{2^t-2-t}) \\
a_2 &= 2(a_1 - 2^{2^t-2-t}) \\
a_3 &= 2a_2 \\
&\vdots \\
a_t &= 2a_{t-1}
\end{aligned}$$

and let

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_t \end{pmatrix}$$

be the solution to

$$\begin{pmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & -1 & \cdots & 0 \\ 0 & 0 & 0 & 2 & \cdots & 0 \\ . & . & . & . & \cdots & . \\ -1 & 0 & 0 & 0 & \cdots & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_t \end{pmatrix} = \begin{pmatrix} 2a_1 \\ 2a_2 \\ 2a_3 \\ \vdots \\ 2a_t \end{pmatrix}.$$

Then the number

$$1\ t\ (t+1)^{2^t-1}\ (2t)^{2^t}\ \cdots\ (2^t-1-t)^{2^{2^t-2-t}-2^{2^t-2-2t}}\ (2^t-2)^{2^{2^t-2-t}}\ (2^t-1)^m\ (2^t)^{x_1}$$

is the start of a divergent RATS sequence of length t in base $(2^t-1)^2+1$.

5. Questions. As we continue to study the game of RATS, we are led to many questions.

First of all, in the first section, the exponent on the next to last number in the sequence for the length 2 divergent sequence in base 19 is 19, the exponent on the next to last number in the sequence for the length 2 divergent sequence in base 37 is 1257 and the exponent on the next to last number in the sequence for the length 2 divergent sequence in base 55 was 80099. Is there a pattern to these exponents? Can an explicit formula be found for these exponents? What about these exponents for divergent sequences of other lengths or in other bases?

Combining Theorem 6 and Lemma 12 gives the following result. That is, in base 3970, there are two different divergent RATS sequences of different lengths, one of length 2 and one of length 6. Can we find other bases with two or more divergent RATS sequences of different lengths?

Theorem 13 proved the existence of length t divergent RATS sequences in base $(2^t-1)^2+1$ where t is a prime or psp. What about the case when t is not a prime and is not a psp?

John Conway has a simple sounding, yet tremendously hard conjecture based on his RATS game in base 10. So far, every positive integer with digits in nondecreasing order (up to 15 digits) which starts a RATS game either cycles or enters Conway's Divergent RATS Sequence. Conway conjectures that this is true for every positive integer. This is still an open problem.

Finally, we conjecture that there is only one divergent sequence of length t for each base.

References

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