THE EIGENVECTORS OF A CERTAIN MATRIX OF BINOMIAL COEFFICIENTS

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1. Introduction.

Define the sequences $\{U_n\}$ and $\{V_n\}$ for all integers n by

$$U_n = pU_{n-1} - qU_{n-2}, \quad U_0 = 0,\ U_1 = 1,$$

$$V_n = pV_{n-1} - qV_{n-2}, \quad V_0 = 2,\ V_1 = p,$$

where p and q are real numbers with $q(p^2-4q) \neq 0$. These sequences were studied originally by Lucas [4], and have subsequently been the subject of much attention.

The Binet forms of U_n and V_n are

$$U_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \text{ and } V_n = \alpha^n + \beta^n,$$

where

$$\alpha = \frac{p + \sqrt{p^2-4q}}{2} \text{ and } \beta = \frac{p - \sqrt{p^2-4q}}{2}$$

are the roots, assumed distinct, of $x^2 - px + q = 0$. We assume further that α/β is not an nth root of unity for any n.

For n greater than or equal to 1, let $S(n)$ be the $n \times n$ matrix defined by

$$S(n) = \begin{pmatrix} 0 & 0 & 0 & \cdots & (-1)^{n-1}\binom{n-1}{n-1}q^{n-1} \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ . & . & . & \cdots & . \\ 0 & 0 & \binom{2}{2}q^2 & \cdots & \binom{n-1}{2}p^{n-3}q^2 \\ 0 & -\binom{1}{1}q & -\binom{2}{1}pq & \cdots & -\binom{n-1}{1}p^{n-2}q \\ \binom{0}{0} & \binom{1}{0}p & \binom{2}{0}p^2 & \cdots & \binom{n-1}{0}p^{n-1} \end{pmatrix}.$$

The element in its ith row and jth column is

$$(-q)^{n-i}\binom{j-1}{j+i-n-1}p^{i+j-n-1}.$$

The matrix $S(n)$ is the general term in a sequence of matrices $\{S(n)\}_{n=1}^{\infty}$, where the first few terms are $S(1) = (1)$,

$$S(2) = \begin{pmatrix} 0 & -q \\ 1 & p \end{pmatrix},$$

and

$$S(3) = \begin{pmatrix} 0 & 0 & q^2 \\ 0 & -q & -2pq \\ 1 & p & p^2 \end{pmatrix}.$$

It can be proved by induction that

$$S^n(2) = \begin{pmatrix} -qU_{n-1} & -qU_n \\ U_n & U_{n+1} \end{pmatrix},$$

and

$$S^n(3) = \begin{pmatrix} q^2U_{n-1}^2 & q^2U_{n-1}U_n & q^2U_n^2 \\ -2qU_{n-1}U_n & -q(U_n^2 + U_{n-1}U_{n+1}) & -2qU_nU_{n+1} \\ U_n^2 & U_nU_{n+1} & U_{n+1}^2 \end{pmatrix},$$

with similar results for the higher order matrices in the sequence $\{S(n)\}_{n=1}^{\infty}$. When $p = -q = 1$, $S(2)$ becomes essentially the Q-Matrix for the Fibonacci numbers. For applications of $S(3)$ and $S(4)$ to the derivation of certain infinite series, and for more background information on these matrices, see [6] and [7].

Carlitz [1] considered the matrix the matrix $S(n)^T$ for the special case $p = -q = 1$. Among other things, he found its eigenvalues and its characteristic polynomial, and stated that its eigenvectors were not evident.

Mahon and Horadam [5] worked with the matrix $S(n)$ for the case $q = -1$ and put forward a conjecture stating its characteristic polynomial. This conjecture was later proved by Duvall and Vaughan [3].

More recently, Cooper and Kennedy [2] considered the matrix $S(n)^T$ and proved a result of Jarden by generalizing the work of Carlitz [1]. If we translate their results to our matrix $S(n)$ they proved, among other things,

(i) The eigenvalues of $S(n)$ are α^{n-1}, $\alpha^{n-2}\beta$, $\alpha^{n-3}\beta^2$, $\ldots$, $\alpha\beta^{n-2}$, β^{n-1}.

(ii) The characteristic equation of $S(n)$ is

$$\sum_{k=0}^{n}(-1)^k q^{(k-1)k/2}\{n,k\}\lambda^{n-k}=0,$$

where

$$\{n,k\}=\begin{cases}1, & \text{for } k=0,n\\ \dfrac{\prod_{i=1}^{n}U_i}{\left(\prod_{i=1}^{k}U_i\right)\left(\prod_{i=1}^{n-k}U_i\right)}, & \text{for } 0<k<n.\end{cases}$$

There remains the question of the eigenvectors of $S(n)$. The purpose of this paper is to answer that question.

2. Eigenvectors of S(n).

Let $0\le k\le n-1$ be a fixed integer,

$$f(x)=(x-\alpha)^k(x-\beta)^{n-1-k}=\sum_{r=0}^{n-1}v_r x^r,$$

and

$$\mathbf{v}=(v_0,v_1,\ldots,v_{n-1})^T.$$

Lemma 1. Let $m\ge 0$. Then,

$$f^{(m)}(x)=m!\frac{f(x)}{(x-\alpha)^m(x-\beta)^m}\sum_{j=0}^{m}\binom{k}{m-j}\binom{n-1-k}{j}(x-\alpha)^j(x-\beta)^{m-j}.$$

Proof. We will use Leibniz's formula for the mth derivative of a product, i.e.,

$$\frac{d^m}{dx^m}g(x)h(x)=\sum_{j=0}^{m}\binom{m}{j}g^{(m-j)}(x)h^{(j)}(x).$$

Using the notation $x^{\underline{n}}$ to denote the falling factorial, it follows that

$$\begin{aligned}f^{(m)}(x)&=\sum_{j=0}^{m}\binom{m}{j}k^{\underline{m-j}}(x-\alpha)^{k-m+j}(n-1-k)^{\underline{j}}(x-\beta)^{n-1-k-j}\\&=m!\frac{f(x)}{(x-\alpha)^m(x-\beta)^m}\sum_{j=0}^{m}\binom{k}{m-j}\binom{n-1-k}{j}(x-\alpha)^j(x-\beta)^{m-j}.\end{aligned}$$

Lemma 2. Let $0 \le m \le n-1$ be a fixed integer. Then,

$$v_{n-1-m} = \sum_{j=0}^{m} (-1)^m \binom{k}{m-j} \binom{n-1-k}{j} \alpha^{m-j} \beta^j$$

and

$$(S(n)\mathbf{v})_{n-1-m} = \sum_{r=m}^{n-1} (-q)^m \binom{r}{m} p^{r-m} v_r.$$

Proof. The first result follows by computing the coefficient of x^{n-1-m} in $f(x)$ by multiplying $(x-\alpha)^k$ times $(x-\beta)^{n-1-k}$. The second result follows by computing the product of $S(n)$ and $\mathbf{v}$.

Theorem.

$$S(n)\mathbf{v} = \alpha^{n-1-k}\beta^k \mathbf{v}.$$

Proof.

$$\begin{aligned}
(S(n)\mathbf{v})_{n-1-m} &= \sum_{r=m}^{n-1} (-q)^m \binom{r}{m} p^{r-m} v_r \\
&= \frac{(-q)^m}{m!} \sum_{r=m}^{n-1} v_r r^{\underline{m}} p^{r-m} \\
&= \frac{(-q)^m}{m!} f^{(m)}(p) \\
&= \frac{(-1)^m (\alpha \cdot \beta)^m}{m!} \frac{(p-\alpha)^k (p-\beta)^{n-1-k}}{(p-\alpha)^m (p-\beta)^m} \\
&\quad m! \cdot \sum_{j=0}^{m} \binom{k}{m-j} \binom{n-1-k}{j} (p-\alpha)^j (p-\beta)^{m-j} \\
&= \alpha^{n-1-k}\beta^k (-1)^m \sum_{j=0}^{m} \binom{k}{m-j} \binom{n-1-k}{j} \beta^j \alpha^{m-j} \\
&= \alpha^{n-1-k}\beta^k \sum_{j=0}^{m} (-1)^m \binom{k}{m-j} \binom{n-1-k}{j} \alpha^{m-j} \beta^j \\
&= \alpha^{n-1-k}\beta^k v_{n-1-m}.
\end{aligned}$$

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